TRINITY COLLEGE FOR WOMEN
NAMAKKAL
Department of Mathematics

PARTIAL DIFFERENTIAL EQUATIONS
23PMA06– Even Semester

HELMHOLTZ OPERATORS

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What is Helmholtz operator ?

The Helmholtz Operator is given b as

$$\nabla^2 F + k^2 F = 0$$

Where $F(x, y, z)$ is a function of x, y and z
k^2 is constant

F should have continuous first and second partial derivatives with respect to x, y and z

Next we will show that the Helmholtz operator in

- **Cylindrical Co-ordinates**
- **Spherical Co-ordinates.**

Cylindrical Co-ordinates

Cylindrical coordinates are **ordered triples in the cylindrical coordinate system** that are used to describe the location of a point.

Spherical Co-ordinates.

Spherical coordinates, also called spherical polar coordinates are a system of curvilinear coordinates that are natural for describing positions on a sphere or spheroid.

Solution in cylindrical coordinates

We consider the Helmholtz operator in cylindrical co-ordinates (r, θ, z) for the function $\psi(r, \theta, z)$

The ∇^2 operator in these co-ordinates is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

We can do the separation $\psi(r, \theta, z) = R(r)\Theta(\theta)Z(z)$.

Using the above expression for the ∇^2operator and the method of separation of variables we can drive the solution of the equation.

After some simplification, we can get the following equations

$$r\frac{d}{dr}\left(r\frac{dp}{dr}\right)+(n^2r^2\text{-}m^2)\mathrm{P} = 0$$

Several comments are in order

- In the first equation, l^2 is chosen to have an exponentially decaying solution.
- In the second equation $-m^2$ is chosen to have a periodic solution

- The third equation in the Bessel equation with argument nr

Among these equations, we get $n^2 = l^2 + k^2$ so there are again two independent parameters among l, m *and* n

Here too, boundary conditions are required to specify the particular solution of the equations.

Solution in spherical co-ordinates

We can use the expression for ∇^2 is spherical co-ordinates (r, θ, φ).

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial \theta} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial^2}{\partial \varphi^2}$$

With it, we can make the separation

$\psi(r, \theta, \varphi) = R(r)\Theta(\theta)Z(z)$ and use the method of separation of variables to get the equation for R, Θ, Z

We get these equations

φ − Equations

$$\frac{d^2\varphi}{d\phi^2} = -m^2\ \varphi(\phi)$$

The constant $-m^2$ is chosen to make $\varphi(\phi)$ a periodic function of ϕ.

Θ-Equations

$$sin^2\theta\,\frac{d^2\Theta}{d(cos\theta)^2}\ \text{-2}\ cos\theta\,\frac{d\Theta}{d(cos\theta)}\left(l(l+1)-\frac{m^2}{sin^2\theta}\right)\Theta = 0$$

This is an associated Legendre equation in the argument $cos\theta$. The term $l(l+1)$ comes from the fact that this equation has non singular solutions only if we have a term $l(l+1)$there.

Applications of Helmholtz operator

- Tsunamis
- Volcanic eruptions
- Medical imaging
- Electromagnetism:

In the science of optics, the Gibbs-Helmholtz equation: Is used in the calculation of change in enthalpy using change in Gibbs energy when the temperature is varied at constant pressure..

THANK YOU

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