TRINITY COLLEGE FOR WOMEN NAMAKKAL
Department of Mathematics

REAL ANALYSIS- I
23PMA02- ODD SEMESTER

SEQUENCE OF FUNCTIONS

Presented by
Mrs. V.GOKILA
Assistant Professor
Department of Mathematics
http://www.trinitycollegenkl.edu.in/

POINTWISE CONVERGENCE OF SEQUENCE OF FUNCTIONS

- The sequence $\{f_n\}$ whose terms are real or complex valued functions having a domain on the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$.

For each x in the domain consider the sequence $\{f_n(x)\}$ whose terms are the corresponding function values.

Let S denote the set of x for which this sequence converges.

- The function f defined by the equation

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{if } x \in S,$$

is called the Limit Function of the sequence $\{f_n\}$, and we say that $\{f_n\}$ converges pointwise to f on the set S.

- The continuity of each f_n at c implies the continuity of the limit function f at c.

ie) $$\lim_{x\to c} f_n(x) = f_n(c)$$

$$\Rightarrow \quad \lim_{x\to c} f(x) = f(c)$$

It can be written as follows;

$$\lim_{x\to c}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to c} f_n(x).$$

EXAMPLES OF SEQUENCES OF REAL-VALUES FUNCTIONS

Example:1

- A sequence of continuous function with a discontinuous limit function

  $f_n(x) = \frac{x^{2n}}{1+x^{2n}}$ if x ∈ ℝ, n=1,2,3,….

Example:2

- The limit of the integral is need not be equal to the integral of the limit

  $f_n(x) = n^2 x(1-x)^n$ if x ∈ ℝ, n=1,2,3,….

Example:3

- The derivative of converges function is need not be converges

  $f_n(x) = \frac{sinnx}{\sqrt{n}}$ if x ∈ ℝ, n=1,2,3,….

UNIFORMLY CONVERGES

A sequence of functions $\{f_n\}$ is said to converge uniformly to f on a set S, if for every ε >0 there exists an N (depending only on ε) such that
N>N $\Rightarrow$ | $f_n(x)$- $f(x)$| < ε for every x in S.
ie) $f_n \to f$ uniformly on S.

UNIFORMLY BOUNDED

- A sequence of functions $\{f_n\}$ is said to bounded on S if there exists a constant M > 0 such that | $f_n(x)$ | ≤ M for all x in S, and all n.

The number M is called a uniform bound for $\{f_n\}$.

- If each individual function is bounded and if $f_n \to f$ uniformly on S, then we say that $\{f_n\}$ is uniformly bounded on S.

Uniform convergence and continuity

Assume that $f_n \to f$ uniformly on S. If each f_n is continuous at a point c of S, Then the limit function f is also continuous at c.

ie) $\lim_{x\to c} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to c} f_n(x)$.

Cauchy condition for uniform convergence

Let $\{f_n\}$ be a sequence of functions defined on a set S. There exists a function f such that $f_n \to f$ uniformly on S iff the following condition is satisfied;

For every ε >0 there exists an N such that m>N and n>N implies $| f_m(x)-f_n(x)| < \varepsilon$ for every x in S.

Weierstrass M-test

Let $\{M_n\}$ be a sequence of non-negative numbers such that $0 \le | f_n(x) | \le M_n$, for n=1,2,3,…. and for every x in S.

Then $\sum f_n(x)$ converges uniformly on S if $\sum M_n$ converges.

EXAMPLE:

- Example for a function which is not uniformly convergence

Let $f_n(x) = x^2$ if $0 \leq x \leq 1$

The convergence is not uniform on [0,1], Since the sequence of continuous functions with discontinuous limit.

PROPERTIES OF UNIFORM CONVERGENCE OF SEQUENCE OF FUNCTIONS:

- Each f_n is bounded ⇒ f is bounded
- Each f_n is Riemann integrable on [a,b] ⇒ f is Riemann integrable on [a,b]
- Each f_n is continuous ⇒ f is continuous
- Each f_n is differentiable ⇒ f is differentiable

BOUNDEDLY CONVERGENT

A sequence of functions $\{f_n\}$ is said to be boundedly convergent on T if $\{f_n\}$ is pointwise convergent and uniformly bounded on T.

THROREM:

Let $\{f_n\}$ be a boundedly convergent sequence on[a,b]. Assume that each $f_n \in \mathbb{R}$ on [a,b], and that the limit function f $\in \mathbb{R}$ on [a,b]. Assume also that there is a partition P of [a,b], say P = $\{X_0, X_1, X_2, X_3, \ldots, X_n\}$ such that on every subinterval [c,d] not containing any of the points X_k, the sequence $\{f_n\}$ converges uniformly to f. Then we have

$$\lim_{n\to\infty} \int_a^b f_n(t)\ \mathrm{dt} = \int_a^b \lim_{n\to\infty} f_n(t)\ \mathrm{dt} = \int_a^b f(t)\ \mathrm{dt}.$$

UNIFORM CONVERGENCE AND DIFFERENTIATION

THEOREM:

Assume that each term of $\{f_n\}$ is a real-valued function having a finite derivative at each point of an open interval (a,b). Assume that for atleast one point X_0 in (a,b) the sequence $\{f_n(X_0)\}$ converges. Assume further that there exists a function g such that $f'_n \to$ g uniformly on (a,b) . Then

a) There exists a function f such that $f_n \to f$ uniformly on (a,b).

b) For each x in (a,b) the derivative f '(x) exists and equals g(x).

MEAN CONVERGENCE

Let $\{f_n\}$ be a sequence of Riemann-integrable functions defined on [a,b]. Assume that $f \in \mathbb{R}$ on [a,b]. The sequence $\{f_n\}$ is said to converge the mean to f on [a,b],

And we write

$$\lim_{n\to\infty} f_n = f \qquad \text{on} \quad [a,b].$$

if

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^2 \, dx = 0.$$

If the inequality $|f(x) - f_n(x)| < \mathcal{E}$ holds for every x in [a,b], then we have $\int_a^b |f(x) - f_n(x)|^2 \, dx \leq \mathcal{E}^2(b-a)$.

Therefore uniform convergence of $\{f_n\}$ to f on [a,b] implies mean convergence.

THANK YOU

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